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Indeterminacy Relations and Simultaneous Measurements in Quantum Theory

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This paper concerns derivations and interpretations of the uncertainty relations. The exclusive validity of the statistical interpretation is called into question. An individualistic interpretation, formulated by means of the concept of unsharp observables, is justified through a model of a joint measurement of position and momentum.

1. INTRODUCTION AND SUMMARY

Although the uncertainty relations (UR) were discovered by Heisenberg (1927) more than 50 years ago, discussion of their interpretation has not ceased. There are a large number of publications on this issue, and we must therefore restrict ourselves to a selection of typical statements. Many textbooks,[2] essentially following Heisenberg's (1927, 1930) original ideas, expound the well-known ("common sense") individualistic interpretation (I) of the UR: according to (I), the inequality

$$\Delta q \cdot \Delta p \geq \tfrac{1}{2}\hbar \qquad (1)$$

determines the limits of the "unsharpnesses" which are connected necessarily with the values of joint position-momentum measurements; thus it is impossible to measure *simultaneously* position $(q, \Delta q)$ and momentum $(p, \Delta p)$ of an *individual* particle with "inaccuracies" $\Delta q, \Delta p$ violating (1). As a justification of (I) there are usually given derivations of (1) through thought experiments such as Heisenberg's slit experiment or γ microscope.

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[2]Compare the books of Jauch (1968), Landau and Lifshitz (1959), Messiah (1970), and Schiff (1968).

On the other hand, a number of authors[3] claim the exclusive validity of the statistical interpretation (S) saying that the values of position *or* momentum measurements made on an *ensemble* of systems in state W are distributed with statistical spreads $\Delta q \equiv \Delta_W Q$ and $\Delta p \equiv \Delta_W P$, respectively, which obey the inequality

$$\Delta_W Q \cdot \Delta_W P \geq \tfrac{1}{2}\hbar \tag{2}$$

This relation is known to be a consequence of the Hilbert space formalism of quantum mechanics. It follows from the commutation relation

$$[Q, P]\varphi = i\hbar\varphi \qquad (\varphi \in \mathrm{dom}(QP) \cap \mathrm{dom}(PQ)) \tag{3}$$

for position and momentum operators Q, P forming a Schrödinger couple.[4]

Interpretation (I) of the UR has been challenged in two ways. Popper (1934), Einstein (cf. Bohr, 1949) and Park and Margenau (1968) try to give examples of joint measurements with *arbitrary* accuracy. On the other side von Neumann (1932) formulates an incommensurability theorem for incompatible[5] observables, and according to Suppes (1961), the strict impossibility of joint measurements of position and momentum can be inferred from the nonexistence of joint probability distributions. Obviously, the two theses are in contradiction with each other, but each of them disproves (I). It appears that the three standpoints concerning the feasibility of joint measurements coexist up to now. In our opinion the confused situation results from two weaknesses in the various argumentations.

First of all, most thought experiments invented to prove or disprove (I) are not discussed strictly quantum mechanically but only in a semiclassical way; thus the possibility of coming into conflict with the formalism of quantum mechanics cannot be excluded and, as a matter of fact, Einstein's and Popper's proposals were shown to be wrong immediately (von Weizsäcker, 1934; Bohr, 1949). There are, of course, some quantum mechanical treatments of measuring processes: von Weizsäcker's (1931) quantum electrodynamical calculation of the γ microscope which could be regarded as an example in favor of (I), and Park's and Margenau's (1968) "historical" joint measurements with arbitrary accuracy. But in these approaches, as in the former cases, a second deficiency must be mentioned. Either the authors give no explicit definitions of a concept of "measurement" or, if they do, their concepts are tacitly assumed to be the only possible ones. The first alternative applies to all discussions of semiclassical thought

[3]See, for example, Margenau (1963), Park (1968), Ballentine (1970), Ludwig (1976), and Gibbins (1981).

[4]The question whether position and momentum observables are better represented by Heisenberg couples than by Schrödinger couples is discussed in some detail in Lahti (1980).

[5]We shall call pairs of observables incompatible if their operators do not commute.

experiments and to von Weizsäcker's work where only intuitive ideas of "measurement" and "accuracy" are presupposed. Von Neumann's (1932) and Lüders' (1951) incommensurability theses as well as the nonexistence of joint probabilities rest upon the concept of (ideal) first-kind measurements which shall be seen to contain strong idealizations in the next section. That Park's and Margenaus's examples represent quite a strange conception of "measurement" is pointed out by Jauch (1974) and by de Muynck et al. (1979).

Thus the question of the tenability of the individualistic interpretation (I) of the UR still seems to be open. To systematize its discussion first of all a precise quantum mechanical formulation of (I) is needed. This can be given by means of the concept of fuzzy, or unsharp observables which has been developed by Davies and Lewis (1970) (who used the name "modified observables") and by Ali and Emch (1974) (who were the first to notice its significance for measuring "unsharpness"). In Section 2 it is argued that this notion just represents those generalizations of the first kind measurements necessary to make possible joint position-momentum measurements. The corresponding definitions of "unsharpness" Δq and Δp are seen to be in good agreement with physicists' intuitive conceptions of real quantum measuring situations. In Section 3 a quantum mechanical model of a joint position-momentum measurement is presented from which the precise meaning and the origin of the "unsharpness" can be read off: they represent measures of the *objective indeterminateness* of position and momentum of a particle after *preparatory measurements* and *must not* be regarded as *measurement inaccuracies* (except for some classical limit). So relation (1) should be called "indeterminacy relation" or, since we shall retain the familiar name "uncertainty relation," one must keep in mind that the Δq, Δp are measures of a kind of *objective* uncertainty or objective *undecidedness.*

With these results it is possible to close a "logical gap" between (2) and (1) which was shown up by Popper (1932) and by Suppes (1961). If one assumes an individualistic[6] instead of a statistical interpretation of quantum mechanical probabilities and thus of (2) then (1) becomes a (logical) consequence of (2): *if* in nature position and momentum of a particle must be indeterminate according to (2) *then* (preparatory) joint measurements necessarily lead to values $(q, \Delta q)$, $(p, \Delta p)$ with unsharpness satisfying (1). This conclusion results from the fact that the trace class operators W appearing in (2) may be interpreted in at least two different ways: first as state operators, and second, as (operator) densities for some joint position momentum observables. Each of these possibilities gives rise

[6]There exists, of course, a probability theory built up operationally by means of a language referring to *single* physical systems. See Mittelstaedt (1981) and Stachow (1981).

to an interpretation of (2), a probabilistic (individualistic or statistical) one referring to sharp (single) measurements and a nonprobabilistic and individualistic one with respect to joint measurements with certain unsharpnesses, that is, (1).

2. TOWARD A CONCEPT OF JOINT MEASUREMENTS

2.1. Incommensurability of Incompatible Observables

The fundamental concepts of quantum mechanics in its standard formulation are those of states and of observables of a physical system S which reflect the procedures of preparing S and of performing measurements on it. While states are described through positive trace class operators $W \in \mathcal{T}_s(\mathcal{H})^+$, observables usually are represented by self-adjoint operators A on the Hilbert space $\mathcal{H}$ of S or, equivalently, by their spectral measures $P^A: E \mapsto P^A(E)$. The expressions

$$p_W^A(E) \equiv \mathrm{tr}[W \cdot P^A(E)] \tag{4}$$

are interpreted as the probabilities of finding a result within the Borel set $E \in \mathcal{B}(\mathbb{R})$ if an A measurement is made on S in state W. A quantum mechanical theory of measurement shows that the measuring processes compatible with this interpretation are (ideal) first-kind measurements.[7] These measurements may be characterized by their effect on the states of S. Let us consider the pure discrete spectrum case,

$$A = \sum_n a_n P_n^A = \sum_{n,\lambda} a_n P[\varphi_{n,\lambda}]$$

then one may distinguish two limiting cases:

$$W \to (W; A)_N = \sum_{n,\lambda} P[\varphi_{n,\lambda}] W P[\varphi_{n,\lambda}]$$

$$\to W_{n_0\lambda_0} = P[\varphi_{n_0,\lambda_0}] W P[\varphi_{n_0,\lambda_0}] \tag{5a}$$

which is von Neumann's nonideal maximal measurement, and

$$W \to (W; A)_L = \sum_n P_n^A W P_n^A \to W_{n_0} = P_{n_0}^A W P_{n_0}^A \tag{5b}$$

Lüders' ideal measurement.

The operators $(W; A)$ correspond to the mixed states of S after the measurement interaction has ceased but before the results are registered.

[7]This has been elaborated on in von Neumann (1932), Süssmann (1958), and Mittelstaedt (1976).

As soon as a value a_{n_0} has been fixed the system is known to be in state $W_{n_0\lambda_0}$ of W_{n_0}. Both, maximal as well as ideal, measurements of two observables A, B can be done jointly if and only if A and B commute. For example, Lüders proved the theorem

$$[A, B]=0 \curvearrowright \forall_W \forall_n \qquad p_W^A(\{a_n\}) = p_{(W;B)_L}^A(\{a_n\})$$

Thus incompatible observables are incommensurable in the sense of first-kind measurements, and we conclude that this notion cannot in fact be employed for a formulation of (I). Yet it is possible to give a nontrivial conception of joint measurements for incompatible observables A and B if we regard first-kind measurements of functions $F(A)$, $G(B)$ as "unsharp" measurements of A and B. There exist pairs of incompatible observables A, B with some commuting functions which could be measured simultaneously. But still this concept is too narrow for our purposes. Since position (or momentum) observable Q has continuous spectrum the only way to perform a first-kind position measurement is by measuring some function

$$F(Q) = \sum_{n=-\infty}^{\infty} q_n Q(E_n) \qquad \left(E_n \cap E_m = \varnothing, \bigcup_n E_n = \mathbb{R}\right)$$

The following theorem (Lenard, 1972; Jauch, 1974; Amrein and Berthier, 1977) forbids any joint measurement of such functions $F(Q)$ and $G(P)$:

Theorem 6. Let $Q(E)$, $P(F)$ be the spectral measures of Q, P; E and F be measurable sets with nonvanishing finite Lebesgue measures $0<\mu(E)\cdot\mu(F)<\infty$. Then

$$Q(E) \wedge P(F) = 0 \tag{6a}$$

$$Q(\mathbb{R}\backslash E) \wedge P(F) = 0 \qquad \text{(for semibounded } E, F) \tag{6b}$$

$$Q(E) \wedge P(\mathbb{R}\backslash F) = 0 \qquad \text{(for semibounded } E, F) \tag{6c}$$

$$Q(\mathbb{R}\backslash E) \wedge P(\mathbb{R}\backslash F) = R \neq 0, \qquad \dim(R\mathscr{H}) = \infty \tag{6d}$$

So the best one can do by means of first-kind measurements is to simultaneously "localize" a particle into complements of, e.g., intervals E, F of position and momentum spectra. Lahti (1980) takes (6a) as an expression of complementarity in the sense of Bohr: measuring instruments which allow a unique determination of position or momentum (in intervals E, F) are mutually exclusive.

The reason for the strict incommensurability of incompatible observables must be seen in the representation of these observables through projection-valued (PV) measures which brings about the existence of first-kind measurements. We call $P^A(E)$ (or E) an *objective property* of S in W if $W = P^A(E)\, WP^A(E)$, i.e. if W is an eigenstate of $P^A(E)$. In this case the results E' ($E \subseteq E'$) of A measurements can be predicted with certainty, $p^A_W(E') = 1$; measurements of the first kind may therefore be called "objectifying" measurements. Theorem 6 means that there are no (nontrivial) objectifying joint measurements of position and momentum. If such measurements exist they must be nonobjectifying, which means that the corresponding concept of observables can no longer be founded on PV measures.

2.2. Introduction of Unsharp Observables

From a frame theory of quantum mechanical measurement the most general description of observables can be inferred (Ludwig, 1976). Probabilities generally are expressed by means of positive operators a with $0 \leq a \leq 1$ called effects,

$$p = \mathrm{tr}[W \cdot a]$$

whereas observables may be represented by normalized positive operator-valued (POV) measures $a: E \mapsto a(E)$ which uniquely define self-adjoint operators $A = \int_{\mathbb{R}} \lambda \, da(E_\lambda)$. Obviously the standard (PV) observables are contained as special cases. Now we have to find a more general class of POV observables for which the above-mentioned obstacles to joint measurements do not occur. We shall give first an intuitive sketch of the demands to be fulfilled by the "new" kind of observables. Then it will become clear that unsharp observables (Ali and Emch, 1974; Ali and Prugovecki, 1976) possess all desired properties.

It was von Neumann,[8] who pointed to the idealizations inherent in first-kind position measurements; the boundaries of position intervals are nothing but *mathematical* points which cannot be determined *physically* with unlimited precision. According to von Neumann there was no need to elaborate this idea of unsharply defined (i.e., "fuzzy") sets at that time. From the above considerations it becomes apparent that it might be just the modification necessary to achieve the desired concept of "unsharpness" in measurements. We get a nice physical argument against the possibility of measuring a property $Q(E)$ (E being an interval) if we remember that it takes infinitely high potential wells to localize a particle within an interval E. In other words, an infinite amount of energy is needed to prevent a bounded particle from tunneling away. So from scattering an object particle at another (measuring) particle localized by means of a *realistic* potential

[8]von Neumann (1932), pp. 222–223, footnote 126.

one can infer only some unsharply defined region of space but never a true Borel interval as the object's measured position.

It is the transition from ordinary sets to fuzzy sets that enables one to describe just this sort of unsharpness. Whereas in ordinary set theory the element relation $x \in E$ can be represented by the characteristic function χ_E of E, $\chi_E(x) = 1$, in the case of a fuzzy set χ_E is replaced by a more general function ν_E with the following properties:

$$0 \leq \nu_E(x) \leq 1$$

for any x, $E \mapsto \tilde{\nu}_x(E) \equiv \nu_E(x)$ is a measure (Prugovecki 1974).

The only difference from a characteristic function χ_E is that ν_E may assume values unequal to 1 or 0. Now we proceed to the corresponding change in the description of observables. A spectral projection $Q(E)$ is defined in configuration space representation by means of the equation

$$(Q(E)\varphi)(q) = \chi_E(q)\varphi(q)$$

Replacing χ_E by ν_E gives a new "effect"

$$(a(E)\varphi)(q) = \nu_E(q)\varphi(q) \tag{7a}$$

where $a: E \mapsto a(E)$ is a POV measure called *unsharp position observable*, or *fuzzy observable*. [The fuzzy sets $\hat{E} = (E, \nu_E)$ need not be explicitly introduced into (7a); instead we shall always use the usual mathematically idealized sets E.] We shall assume that to $\nu_E(q)$ there corresponds a density function f_q such that

$$\nu_E(q) = \int_E dq' \hat{f}_q(q') \tag{7b}$$

Furthermore, the functions $\hat{f}_q$ shall be supposed to be essentially the same for all q, i.e.,

$$\hat{f}_q(q') = \hat{f}(q' - q) \tag{7c}$$

which means that the position-measuring device corresponding to the fuzzy observable a works equally well in all regions of space. Then we may express (7a–c) in the following form:

$$a(E) = \int_{\mathbb{R}} \nu_E(q) Q(dq) = \int_{\mathbb{R}} dq f(q) Q(E+q) \equiv Q_f(E) \tag{8a}$$

$$\nu_E(q) = \int_E dq' f_{q'}(q) = (f * \chi_E)(q) \tag{8b}$$

$$f_{q'}(q) \equiv f(q - q') := \hat{f}(q' - q) \equiv \hat{f}_q(q') \tag{8c}$$

The operators $a(E)$ appear as weighted means of all the spectral projections $Q(E+q)$ generated from $Q(E)$ by a translation. The functions $f_q:\mathbb{R}\to\mathbb{R}$ should obey the following postulates:

$$f_q(q')\geq 0 \tag{9a}$$

$$f_q(q')=f_0(q'-q)\equiv f(q'-q) \tag{9b}$$

$$\int_{\mathbb{R}} dq' f_q(q')=1 \tag{9c}$$

$$\langle Q\rangle_{f_q}\equiv\int_{\mathbb{R}} dq'\, q' f_q(q')=q \tag{9d}$$

$$(\Delta_f Q)^2\equiv\langle Q^2\rangle_{f_q}-\langle Q\rangle_{f_q}^2\equiv(\Delta q)^2<\infty \tag{9e}$$

Conditions (9a)-(9c) are evident from the introduction of the f_q; by (9d) and (9e) we select such measures $E\mapsto\nu_E(q)$ for which the functions $q\mapsto \nu_E(q)\equiv(f*\chi_E)(q)$ are roughly concentrated around E. The POV measure $E\mapsto Q_f(E)$ defines a unique self-adjoint operator

$$Q_f\equiv\int_{\mathbb{R}} qQ_f(dq)=Q-\langle Q\rangle_f 1=Q$$

which turns out to coincide with the "old" position operator (See also Schroeck, 1978). The spectral (PV) measure is obtained by setting $f(q)=\delta(q)$. Another useful form of $Q_f(E)$ is the following:

$$\begin{aligned} Q_f(E)&=(\chi_E*f)(Q)=\int_E dq\,\mathcal{F}(q)\\ \mathcal{F}(q)&=\int_{\mathbb{R}} Q(dq')f_q(q')\equiv f_q(Q)\end{aligned} \tag{10}$$

In the following we assume f to be such that $\mathcal{F}(q)$ is bounded for all q.

A measurement of the POV observable Q_f shall be described by its influence on the state of the object:

$$\begin{aligned} &\mathcal{M}_f: E\mapsto\mathcal{M}_f(E)\\ &\mathcal{M}_f(E): W\mapsto\mathcal{M}_f(E)(W)=\int_E dq\, A_q W A_q^+\end{aligned} \tag{11}$$

A_q linear, bounded, and such that $W\geq 0$ implies $\mathcal{M}_f(E)(W)\geq 0$.

The connection with Q_f is given by the probability formula

$$p_W^f(E)\equiv\text{tr}[W\cdot Q_f(E)]=\text{tr}[\mathcal{M}_f(E)(W)] \tag{12a}$$

which implies

$$Q_f(E) = \int_E dq\, A_q^+ A_q \tag{12b}$$

$$\mathcal{F}(q) = A_q^+ A_q \tag{12c}$$

As we shall see below the individualistic interpretation (I) of the UR refers to certain preparatory measurements. A measurement of the unsharp observable Q_f will be called *preparatory* if after registration of an "unsharp point" (q, f_q) the system will be found in a (normalized) final state W_q with

$$\langle Q\rangle_{W_q} \simeq q = \langle Q\rangle_{f_q}, \qquad \Delta_{W_q} Q \simeq \Delta q = \Delta_f Q \qquad (W \in \mathcal{S}) \tag{13}$$

and a *strict preparatory* measurement leads to a final state W_q obeying

$$\langle q'|W_q|q'\rangle = f_q(q') \qquad (W \in \mathcal{S}) \tag{14}$$

In both cases the possibility is included that such final states W_q can be reached only from certain restricted classes $\mathcal{S} \subseteq \mathcal{T}_s(\mathcal{H})^+$ of initial states W. (In general a measuring instrument is suited for certain situations, i.e., preparations of the objects, only.) By an unsharp point (q, f_q) we mean a set $E_q = [q - \delta q/2, q + \delta q/2]$ with

$$\begin{gathered} \delta q \ll \Delta q \qquad \text{such that} \\ Q_f(E_q) \simeq \mathcal{F}(q)\delta q, \qquad \mathcal{M}_f(E_q)(W) \simeq A_q W A_q^+ \delta q \equiv \tilde{W}_q \end{gathered} \tag{15}$$

The approximate equalities are to be understood in the sense of strong continuity (for $\delta_q \to 0$) and they hold if the map $q \to A_q$ is strongly continuous. From (10) and (12c) it can be shown that A_q has a representation

$$A_q = U_q \int_{\mathbb{R}} Q(dq')\phi(q'-q), \qquad U_q^* = U_q^{-1}, \qquad |\phi(q)|^2 = f(q) \tag{16}$$

Then condition (15) is equivalent to the following one [cf. (11)]:

$$\int_{E_q} dq'\, \phi(\tilde{q} - q')\phi^*(\tilde{q}' - q') \simeq \delta q\, \phi(\tilde{q} - q)\phi^*(\tilde{q}' - q) \qquad (\text{for all } \tilde{q}, \tilde{q}' \in \mathbb{R})$$

which can be satisfied by a suited δq if $\phi(q)$ has a bounded derivative.

It follows that there exists no strict preparatory measurement for all initial states, $\mathcal{S} = \mathcal{T}_s(\mathcal{H})^+$: according to (14) the final state W_q must be independent of the initial state W, so because of (15) A_q and $\mathcal{F}(q)$ must take on the form

$$A_q = |\varphi_q\rangle\langle\psi_q|, \qquad \mathcal{F}(q) = |\psi_q\rangle\langle\psi_q|$$

which is incompatible with (16) and (10). There is, however, a certain restricted class $\mathcal{S} \subset \mathcal{T}_s(\mathcal{H})^+$ of initial states which lead to strict preparatory measurements (see Section 3.1 below).

We have tried to give an intuitively convincing "derivation" of the concept of unsharp observables. The reader may convince himself that this notion fits very well with the usual intuitive ideas about unsharp localization. In his famous 1927 paper Heisenberg takes Gaussians as representations of the final states of a position measurement. Lamb (1969) gives a nice operational description of a position measurement which should easily be expressible in terms of observables, too. Another suggestive example is Wootters' and Zurek's (1979) treatment of the double-slit experiment. It appears that unsharp observables correspond to real measuring situations; therefore we suggest to call the measurement of an unsharp (fuzzy) observable a *real measurement*, in contrast to the ideal first-kind measurements. (The term *unsharp measurement* seems to us quite misleading since as we have argued above the "unsharpness" inherent in the observables may not be confused with measurement errors.)

The introduction of unsharp observables implies an extension of the usual quantum mechanical formalism which in our opinion is fully justified from the fact that it opens the possibility to describe "real" measurements. There are known at least three situations in quantum theory where a concept of "unsharpness" is indispensable. First, the strict incommensurability of incompatible observables may be relaxed only by means of unsharp observables. Second, Wigner (1952) and Araki and Yanase (1960) showed that the possibility of first-kind spin measurements is in contradiction with conservation of angular momentum; the "measurement errors" introduced by these authors to resolve the contradiction are nothing but unsharpnesses of some fuzzy spin observables (Prugovecki, 1977). The third instance lies in the domain of relativistic quantum mechanics: in order to define a covariant localization concept for relativistic particles one has to give up the PV measures (Jauch and Piron, 1967) where the notion of unsharp observables leads to such a concept in a natural way (Ali and Emch, 1974; Prugovecki, 1976a, 1981).

We shall concentrate on the first point, the possibility of joint measurements of unsharp position and momentum.

2.3. Joint Measurements of Unsharp Observables

A joint measurement of some observables will be explained as a measurement of a joint observable. A *joint position-momentum observable* $a_{f,g}$ is defined as a POV measure on phase space $\Gamma = \mathbb{R}^2$

$$a_{f,g}(\Delta) = \int_\Delta dq\, dp\, \mathscr{F}(q, p) \tag{17a}$$

with a continuous positive phase space distribution

$$(q, p) \mapsto \mathscr{F}(q, p), \qquad 0 \leq \mathscr{F}(q, p)$$

bounded and with marginal observables

$$a_{f,g}(E\times\mathbb{R})=Q_f(E),\ a_{f,g}(\mathbb{R}\times F)=P_g(F) \tag{17b}$$

being unsharp position and momentum observables [obeying (8) and (9)], respectively. The measures

$$\Delta\mapsto \mathrm{tr}[W\cdot a_{f,g}(\Delta)]$$

are positive-definite phase space measures and thus may be interpreted as joint probability distributions. Ali and Prugovecki (1977) proved the following theorem: to a given pair of unsharpness measures Δq, Δp there exists a joint position-momentum observable $a_{f,g}$ with $\Delta_f Q=\Delta q$, $\Delta_g P=\Delta p$ if and only if Δq, Δp obey the uncertainty relation (1), i.e.,

$$\Delta q\cdot\Delta p\geq\tfrac{1}{2}\hbar$$

In this case there exists a trace class operator W_0 such that

$$\mathscr{F}(q,p)=(2\pi\hbar)^{-1}\exp\left(\frac{i}{\hbar}pQ\right)\exp\left(-\frac{i}{\hbar}qP\right)W_0\exp\left(\frac{i}{\hbar}qP\right)\exp\left(-\frac{i}{\hbar}pQ\right)$$

$$\equiv(2\pi\hbar)^{-1}W_{qp} \tag{18a}$$

$$f_q(q')=\langle q'|W_{qp}|q'\rangle=\langle q'-q|W_0|q'-q\rangle \tag{18b}$$

$$g_p(p')=\langle p'|W_{qp}|p'\rangle=\langle p'-p|W_0|p'-p\rangle \tag{18c}$$

This construction enables us to overcome the obstacle given by (6):

$$0<a_{f,g}(E\times F)\leq Q_f(E),\qquad 0<a_{f,g}(E\times F)\leq P_g(F)$$

that is, there exists a nontrivial (positive) lower bound to the marginal observables.

Some joint observables $a_{f,g}$ have another advantage over the standard PV observables Q, P: for certain states ψ_0, $W_0=P[\psi_0]$ leads to an informationally complete $a_{f,g}$ (Ali and Prugovecki, 1977), that is, $\mathrm{tr}[W_1\cdot a_{f,g}(\Delta)]=\mathrm{tr}[W_2\cdot a_{f,g}(\Delta)]$ [for all $\Delta\in B(\Gamma)$] implies $W_1=W_2$. By such joint measurements the initial states can be uniquely determined, which is not possible even from the combined statistics of sharp $(Q(E), P(F))$ position and momentum measurements. As Ali and Prugovecki called it, standard quantum mechanics is redundant in the sense that it contains classes of indistinguishable different states which redundancy vanishes after the introduction of (joint) unsharp observables. This fact again shows that fuzziness cannot properly be interpreted as (subjective) measurement error.

From theorem (6) together with (18b,c) we learn that not both of the functions f, g may have bounded supports, so joint position-momentum measurements always are nonobjectifying as we had expected (Section 2.1). It appears that in the case of preparatory joint measurements (see below) the $\Delta q, \Delta p$ are measures of the *objective undecidedness* of the measured observables in the final states and that—as we mentioned in the introduction—the uncertainty relation is an "indeterminacy relation". Thus we have established the formulation of the *individualistic interpretation* (I) of the UR which we were looking for. In the next section we shall check this interpretation by means of a theory of measurement which makes it plausible to regard measurements of observables $a_{f,g}$ as joint position and momentum measurements.

To conclude this section we still have to define the notions of (strict) preparatory joint measurements. A joint position-momentum measurement is a measurement of $a_{f,g}$, that is, a positive phase space measure:

$$\begin{aligned} &\mathcal{M}_{f,g}: \Delta \mapsto \mathcal{M}_{f,g}(\Delta) \\ &\mathcal{M}_{f,g}(\Delta): W \mapsto \mathcal{M}_{f,g}(\Delta)(W) = \int_\Delta dq\, dp\, A_{qp} W A_{qp}^+ \end{aligned} \tag{19}$$

with linear bounded A_{qp} such that $\mathcal{M}_{f,g}(\Delta)(W) \geq 0$ for $W \geq 0$. The probability of getting a result Δ is

$$p_W^{f,g}(\Delta) \equiv \mathrm{tr}[W \cdot a_{f,g}(\Delta)] = \mathrm{tr}[\mathcal{M}_{f,g}(\Delta)(W)] \tag{20a}$$

which leads to

$$a_{f,g}(\Delta) = \int_\Delta dq\, dp\, A_{qp}^+ A_{qp} \tag{20b}$$

$$\mathcal{F}(q, p) = A_{qp}^+ A_{qp} \tag{20c}$$

In analogy to the case of single observables we consider the registration of unsharp points $((q, f_q), (p, g_p))$, or better, $E_q \times F_p$ for which [cf. (15)]

$$\begin{aligned} &a_{f,g}(E_q \times F_p) \simeq \mathcal{F}(q, p) \cdot (\delta q\, \delta p) \\ &\mathcal{M}_{f,g}(E_q \times F_p)(W) \simeq A_{qp} W A_{qp}^+ \cdot (\delta q\, \delta p) \equiv \tilde{W}_{qp} \end{aligned} \tag{21}$$

The approximate equalities are guaranteed through strong continuity of the map $(q, p) \to A_{qp}$. $\mathcal{M}_{f,g}$ is called a *preparatory* measurement if for a certain class $\mathcal{S}$ of initial states it leads to (normalized) final states W_{qp} obeying

$$\begin{aligned} &\langle Q \rangle_{W_{qp}} \simeq q = \langle Q \rangle_{f_q}, \qquad \Delta_{W_{qp}} Q \simeq \Delta q = \Delta_f Q \\ &\langle P \rangle_{W_{qp}} \simeq p = \langle P \rangle_{g_p}, \qquad \Delta_{W_{qp}} P \simeq \Delta p = \Delta_g P \end{aligned} \qquad (W \in \mathcal{S}) \tag{22}$$

A *strict preparatory* measurement gives W_{qp} which possesses the following property:

$$\langle q' | W_{pq} | q' \rangle = f_q(q'), \qquad \langle p' | W_{qp} | p' \rangle = g_p(p') \qquad (W \in \mathcal{S}) \tag{23}$$

According to (21) and (23) a strict preparatory measurement for $\mathcal{S} = \mathcal{T}_s(\mathcal{H})^+$ is possible only if

$$A_{qp} = |\varphi_{qp}\rangle\langle\psi_{qp}|(2\pi\hbar)^{-1/2}, \qquad W_{qp} = P[\varphi_{qp}]$$
$$\mathcal{F}(q, p) = (2\pi\hbar)^{-1}P[\psi_{qp}]$$

for which we get

$$f_q(q') = |\varphi_{qp}(q')|^2 = |\psi_{qp}(q')|^2, \qquad g_p(p') = |\tilde{\varphi}_{qp}(p')|^2 = |\tilde{\psi}_{qp}(p')|^2$$

Thus only observables generated through (18) by *pure* states $W_0 = P[\psi_0]$ allow strict preparatory measurements.

In the present context we may restrict ourselves to (strict) preparatory measurements[9] upon which the interpretation (I) of the UR is based.

3. MODEL OF A JOINT POSITION-MOMENTUM MEASUREMENT

3.1. Real Position Measurement

First we shall study a model of a "real" position measurement illustrating the origin of the unsharpness (i.e., function f) inherent in the unsharp observables Q_f. The combined application of two such devices for position and momentum will then be seen to lead to a joint measurement of both observables.

The position measuring apparatus $\mathcal{A}$ shall be divided into a microscopic part M and a macroscopic part M', $\mathcal{A} = M \& M'$. The measuring process consists of an interaction between object system S and M followed by a measurement on M by means of M'. Only the S–M part is analyzed quantum mechanically, whereas the M–M' part is treated as an ideal first-kind measurement. The system $S \& M$ is represented by the tensor product Hilbert space $\mathcal{H}(S) \otimes \mathcal{H}(M)$ of the constituents S and M and its time evolution is generated by the Hamiltonian

$$H = H_S + H_M + H_I$$

where

$$H_S = \frac{P^2}{2m_S} + V, \qquad H_M = \frac{P_M^2}{2m_M}, \qquad H_I = \lambda Q \otimes P_M \delta(t) \qquad (\lambda > 0);$$

The δ function has been introduced as a simplified description of the impulsive character of the measurement. The change of state of S & M

[9] A general classification of measurements has been given in a paper by P. Lahti and the present author [Busch and Lahti (1984)].

from time $t=0_-$ to $t=0_+$ is

$$|\Psi'\rangle \equiv |\Psi(0_+)\rangle = U|\Psi(0_-)\rangle \equiv U|\Psi\rangle$$

$$U \equiv U(0_+, 0_-) = \exp\left\{-\frac{i}{\hbar}\int_{0_-}^{0_+} H\,dt\right\} = \exp\left\{-\frac{i}{\hbar}\lambda Q\otimes P_M\right\}$$

With

$$|\Psi\rangle = |\varphi\rangle \otimes |\Phi\rangle$$

we get

$$|\Psi'\rangle = \int_{\mathbb{R}} dq \int_{\mathbb{R}} dX\, \varphi(q)\Phi(X-\lambda q)|q\rangle\otimes|X\rangle$$

($|q\rangle$ and $|X\rangle$ are the generalized eigenstates of position operators Q and Q_M, respectively.) Immediately after the interaction has finished a first-kind position measurement on M (by means of M') may lead to the objectification of some M property $Q_M(\tilde{E})$; the corresponding S & M state reduction is

$$|\Psi'\rangle \xrightarrow{\tilde{E}} |\tilde{\Psi}(E)\rangle = [I\otimes Q_M(\tilde{E})]|\Psi'\rangle$$

$$= \lambda \int_{\mathbb{R}} dq \int_E dq'\, \varphi(q)\Phi(\lambda(q'-q))|q\rangle\otimes|\lambda q'\rangle$$

(where we have substituted $X=\lambda q'$, $\tilde{E}=\lambda E$). We get the effective change of state of S by reducing the projection operator $P[\tilde{\Psi}(E)]$ to the Hilbert space $\mathscr{H}(S)$:

$$P[\varphi] \to \tilde{W}(E) = \int_E dq A_q P[\varphi] A_q^+ \tag{24a}$$

$$A_q = \sqrt{\lambda} \int_{\mathbb{R}} \Phi(\lambda(q-q'))Q(dq') \tag{24b}$$

$$(A_q\varphi)(q') = \sqrt{\lambda}\,\varphi(q')\phi(\lambda(q-q')) \equiv \varphi_q(q') \equiv \varphi(q')\psi_q(q') \tag{24c}$$

The measurement defined through (24) turns out to be a measurement of an unsharp position observable Q_f. This follows directly by comparison of (24b) with (16) or by application of (12):

$$a(E) = \int_E dq\, \mathscr{F}(q) = \int_{\mathbb{R}} (f*\chi_E)(q)Q(dq) \equiv Q_f(E) \tag{25a}$$

$$f_q(q') = \lambda|\Phi(\lambda(q-q'))|^2 = |\psi_q(q')|^2 \equiv f(q'-q) \tag{25b}$$

The preparation Φ of M must be such that the distribution function f in (25b) fulfils all postulates (9). Since in the (pure) state Φ the M positions are indeterminate within the support of $\Phi(X)$ the model explains the origin of the *indeterminacy measures*

$$\Delta q \equiv \Delta_f Q = \frac{1}{\lambda}\Delta_\Phi Q_M > 0 \tag{25c}$$

from the preparation of the measuring device. The magnitude of Δq may be varied via M preparation (Φ) or interaction (λ).

Before proceeding the following remark seems appropriate. The use of first-kind measurements (on M) in our model appears inconsistent with our approach because we had started by providing arguments against the feasibility of those sort of measurements; thus we should better replace this sharp M position measurement by a measurement of some unsharp M position observable. Yet, as our model shows the idealized (sharp) measurements on M are sufficient to produce fuzziness on the object level. Introduction of unsharp observables on the M level would only give rise to second-order corrections of the S position unsharpness (25c). Therefore, we shall retain that idealization as a mathematical simplification.

Now we have to specify the significance of the indeterminacy measures $\Delta_f Q$ for the object after a preparatory measurement. It is easy to characterize a class $\mathcal{S} \subset \mathcal{T}_S(\mathcal{H})^+$ of initial states which allows (strict) preparatory measurements in the sense of (13) or (14). The state change resulting from registration of an unsharp point (q, f_q) is

$$\varphi(q') \to (\delta q)^{1/2}\varphi_q(q') = (\delta q)^{1/2}\varphi(q')\psi_q(q')$$

[cf. equation (15)]. These final states $\varphi_q(q')$ (for all q) become approximately independent (up to a factor) of the initial state if $\varphi(q')$ may be considered almost constant where the $\psi_q(q')$ appreciably differ from zero, and we may define the class $\mathcal{S}$ as the set of states $P[\varphi]$ with

$$\varphi_q(q') = \varphi(q')\psi_q(q') \simeq \varphi(q)\psi_q(q') \qquad \text{(for all } q, q') \tag{26}$$

The normalized final states $W_q \simeq P[\psi_q]$ obey conditions (14) for strict preparatory measurements. For arbitrary registrations E we have

$$P[\varphi] \to \tilde{W}(E) \simeq \int_E dq\, |\varphi(q)|^2 P[\psi_q] \tag{27a}$$

with probability

$$p_E \equiv p^f_\varphi(E) = \mathrm{tr}[\tilde{W}(E)] \simeq \int_E dq |\varphi(q)|^2 \tag{27b}$$

From this the significance of (26) becomes clear: the class $\mathcal{S}$ of *initial* states $P[\varphi]$ has been chosen such that the distribution f cannot be distinguished from the δ distribution corresponding to sharp position measurements: $p_\varphi^f(E) \simeq p_\varphi^Q(E)$. Yet the difference becomes noticeable from the position uncertainty in the *final* states $W(E)$

$$\begin{aligned}(\Delta_{W(E)}Q)^2 \simeq (\Delta_{\varphi_E}Q)^2 + (\Delta_f Q)^2 \\ \varphi_E \equiv p_E^{-1/2} Q(E)\varphi, \qquad W(E) = p_E^{-1}\tilde{W}(E)\end{aligned} \tag{28}$$

The "localization" of the object into E is uncertain in an objective sense, i.e., *indeterminate* to an amount $\Delta_f Q$. It follows that, contrary to ideal first-kind measurements, real (preparatory) measurements generally are not "strongly" repeatable but only "weakly" repeatable (see footnote 9): one can at best make predictions with probabilities near 1 but not exactly equal to 1. In the above situation [class $\mathcal{S}$, equation (26)] we have to take a "large" set E_ε containing E in order to get a probability

$$p^f_{W(E)}(E_\varepsilon) > 1 - \varepsilon \qquad (0 < \varepsilon \ll 1) \tag{29}$$

for a repeated position measurement.

As repeatability is a consequence of predictability it follows that real measurements are at best "weakly" predictable [in the sense of (29)]. There exists a class of initial states $P[\varphi]$ for which our real position measurement is weakly predictable and weakly repeatable. Let $q_0 = \langle Q\rangle_\varphi$, $\Delta_\varphi Q \ll \Delta_f Q$ so that the functions $\psi_q(q')$ [equation (24c)] may be considered constant where $\varphi(q)$ is not negligibly small. Then we may approximate

$$\varphi_q(q') \equiv \varphi(q')\psi_q(q') \simeq \varphi(q')\psi_q(q_0) \tag{30}$$

and we get

$$P[\varphi] \to \tilde{W}(E) \simeq P[\varphi] \int_E dq\, |\psi_q(q_0)|^2$$

$$p_\varphi^f(E) = \mathrm{tr}[\tilde{W}(E)] \simeq \int_E dq\, f_q(q_0)$$

This probability is almost equal to 1 if $g(q) := f_q(q_0) = f(q_0 - q)$ is concentrated on $E_0 = (q_0 - \delta, q_0 + \delta)$. States φ obeying (30) remain practically unchanged by a real position measurement. Yet, from a result $E \supset E_0$ one cannot get any information about the "actual" value q_0, one only knows that it must lay almost inside E. Although the unsharpness $\Delta_f Q$ originates from some indeterminateness the uncertainty in the determination of q_0 from a result E is of a subjective nature. Thus the name "confidence" function for f [originally used by Prugovecki (1976b)] seems appropriate in this "classical" situation (30) only, whereas in the case of preparatory measurements (26) one should speak of "indeterminacy measure."

It seems typical of real measuring devices that they are applicable to certain restricted classes of situations only, preparatory measurements requiring conditions (26) and "classical" measurements requiring (30).

3.2. Joint Position-Momentum Measurement

We are going to deal with the question what will happen if we combine two measuring devices for unsharp position Q_f and unsharp momentum P_g, respectively. As we have seen, the Hamiltonian

$$H_I^{(Q)} = \lambda Q \otimes P_1 \delta(t) \qquad (\lambda > 0)$$

can be used to construct a model of a real position (Q_{f^0}) measurement; similarly the interaction

$$H_I^{(P)} = \mu P \otimes P_2 \delta(t) \qquad (\mu > 0)$$

between the object system S and a second measuring particle M_2 leads to a real measurement of unsharp momentum P_{g^0}. The preparations Φ_1, Φ_2 of M_1, M_2 give rise to unsharp points (q, f_q^0), (p, g_p^0) with

$$\begin{aligned} f_q^0(q') &= \lambda |\Phi_1(\lambda(q-q'))|^2, \qquad \Delta_{f^0} Q = \frac{1}{\lambda}\Delta_{\Phi_1} Q_1 \\ g_p^0(p') &= \mu |\Phi_2(\mu(p-p'))|^2, \qquad \Delta_{g^0} P = \frac{1}{\mu}\Delta_{\Phi_2} Q_2 \end{aligned} \tag{31}$$

From theorem (18) we should expect that a combination of both apparatuses would lead to a joint measurement only if preparations Φ_1, Φ_2 obey certain conditions. But as will be seen there are no serious restrictions at all.

We start with the Hamiltonian

$$H_I \equiv H_I^{(Q,P)} = \{\lambda Q \otimes P_1 \otimes 1 + \mu P \otimes 1 \otimes P_2\}\delta(t) \tag{32}$$

The initial state at $t = 0_-$ of the combined system S & M_1 & M_2,

$$|\Psi\rangle = |\varphi\rangle \otimes |\Phi_1\rangle \otimes |\Phi_2\rangle$$

is transformed by

$$U = U(0_+, 0_-) = \exp\left\{-\frac{i}{\hbar}\lambda Q \otimes P_1 \otimes I - \frac{i}{\hbar}\mu P \otimes I \otimes P_2\right\} \tag{33a}$$

into the final state at $t = 0_+$,

$$|\Psi'\rangle = U|\Psi\rangle$$

From the identity

$$\exp(A+B) = \exp(A)\cdot\exp(B)\cdot\exp(-\tfrac{1}{2}[A, B])$$

$$(\text{for } [A,[A,B]] = [B,[A,B]] = 0) \tag{33b}$$

we get the following useful form of U:

$$U = \exp\left\{-\frac{i}{\hbar}\lambda Q\otimes P_1\otimes I\right\}\cdot\exp\left\{-\frac{i}{\hbar}\mu P\otimes I\otimes P_2\right\}$$

$$\cdot\exp\left\{\frac{\lambda\mu}{2\hbar^2}[Q, P]\otimes P_1\otimes P_2\right\} \tag{33c}$$

We denote by Q_1, Q_2 the position operators of M_1, M_2; further let $|X_i\rangle$, $|\Pi_i\rangle$ be the eigenstates of Q_i, P_i; $\Phi_i(X_i)\equiv\langle X_i|\Phi_i\rangle$, $\tilde{\Phi}_i(\Pi_i)\equiv\langle\Pi_i|\Phi_i\rangle$, $|q\rangle\otimes|X_1\rangle\otimes|X_2\rangle\equiv|qX_1X_2\rangle$, etc. Then the following holds:

$$\begin{aligned}\Psi'(q, X_1, X_2) &\equiv \langle qX_1X_2|\Psi'\rangle\\ &= \int_{\mathbb{R}}\frac{d\Pi_2}{(2\pi\hbar)^{1/2}}\exp\left(\frac{i}{\hbar}X_2\Pi_2\right)\\ &\quad\cdot\varphi(q-\mu\Pi_2)\Phi_1\left(X_1-\lambda q+\frac{\lambda\mu}{2}\Pi_2\right)\tilde{\Phi}_2(\Pi_2)\end{aligned} \tag{34}$$

As in the case of the single measurements there are performed ideal first-kind measurements of $Q_1(\tilde{E})$ and $Q_2(\tilde{F})$ at $t=0_+$:

$$|\Psi'\rangle\to|\tilde{\Psi}(E, F)\rangle = [I\otimes Q_1(\tilde{E})\otimes Q_2(\tilde{F})]|\Psi'\rangle$$

The final state of S is

$$\tilde{W}(E, F) = \int_E dq\int_F dp\, A_{qp}P[\varphi]A_{qp}^+ \quad (\lambda E\equiv\tilde{E}, \mu F\equiv\tilde{F}) \tag{35a}$$

$$\begin{aligned}A_{qp} &= \left(\frac{\lambda}{\mu}\right)^{1/2}\int_{\mathbb{R}}dq'\int_{\mathbb{R}}dq''\frac{1}{(2\pi\hbar)^{1/2}}\exp\left[\frac{i}{\hbar}p(q'-q'')\right]\\ &\quad\cdot\Phi_1\left(\lambda(q-q')+\frac{\lambda}{2}(q'-q'')\right)\cdot\tilde{\Phi}_2\left(\frac{1}{\mu}(q'-q'')\right)|q'\rangle\langle q''|\end{aligned} \tag{35b}$$

$$A_{qp}|\varphi\rangle\equiv|\varphi_{qp}\rangle = (\lambda\mu)^{1/2}\int_{\mathbb{R}}dq'|q'\rangle\Psi'(q', \lambda q, \mu p) \tag{35c}$$

The corresponding observable a is given through equation (20); its marginal observables are of the type of unsharp position and momentum observables, respectively:

$$a(E\times\mathbb{R})=Q_f(E),\qquad a(\mathbb{R}\times F)=P_g(F) \tag{36}$$

$$\begin{aligned}
f_q(q')&=2\pi\hbar\langle q'|\mathscr{F}(q,p)|q'\rangle\\
&=\frac{\lambda}{\mu}\int_{\mathbb{R}}dq''\left|\Phi_1\left(\lambda(q-q')-\frac{\lambda}{2}q''\right)\right|^2\cdot\left|\tilde{\Phi}_2\left(\frac{1}{\mu}q''\right)\right|^2\\
g_p(p')&=2\pi\hbar\langle p'|\mathscr{F}(q,p)|p'\rangle\\
&=\frac{\mu}{\lambda}\int_{\mathbb{R}}dp''\left|\tilde{\Phi}_1\left(\frac{1}{\lambda}p''\right)\right|^2\cdot\left|\Phi_2\left(\mu(p-p')+\frac{\mu}{2}p''\right)\right|^2
\end{aligned} \tag{37}$$

It is easy to prove that f, g fulfil conditions (9). We restrict ourselves to (9d) and (9e). From (37) it follows

$$\begin{aligned}
\langle Q\rangle_{f_q}&=q-\frac{1}{\lambda}\langle Q_1\rangle_{\Phi_1}-\mu\langle P_2\rangle_{\Phi_2}=q\\
\langle P\rangle_{g_p}&=p-\frac{1}{\mu}\langle Q_2\rangle_{\Phi_2}-\lambda\langle P_1\rangle_{\Phi_1}=p
\end{aligned} \tag{38}$$

The expectation values $\langle Q_i\rangle_{\Phi_i}$ vanish since the preparations had been assumed to fulfil (9d); the $\langle P_i\rangle_{\Phi_i}$ have been made zero by choice. Similarly (9e) is implied from the corresponding relations for f^0, g^0 in (31) if we additionally demand $\Delta_{\Phi_i}P_i<\infty$:

$$\begin{aligned}
(\Delta_f Q)^2&=\frac{1}{\lambda^2}(\Delta_{\Phi_1}Q_1)^2+\frac{\mu^2}{4}(\Delta_{\Phi_2}P_2)^2<\infty\\
(\Delta_g P)^2&=\frac{1}{\mu^2}(\Delta_{\Phi_2}Q_2)^2+\frac{\lambda^2}{4}(\Delta_{\Phi_1}P_1)^2<\infty
\end{aligned} \tag{39}$$

Thus we have established the following important result: the simultaneous application of two instruments M_1, M_2 suited for real position or momentum measurements leads to a joint measurement if each of the states Φ_1, Φ_2 may be considered a final state of some preparatory joint position-momentum measurement; that is, Φ_i may be characterized by unsharp position and momentum points. The observable $a=a_{f,g}$ is a joint position-momentum observable. It is from equation (39) that we are justified to interpret the unsharpness $\Delta_f Q$, $\Delta_g P$ as *measures of indeterminateness*. The representations (37) of f, g through some trace class operator

$$W_{qp}=(2\pi\hbar)\mathscr{F}(q,p),\qquad \mathrm{tr}[W_{qp}]=1 \tag{40}$$

guarantees that the UR is valid for $\Delta_f Q$, $\Delta_g P$, independent of the particular values of the $\Delta_{f^0} Q$, $\Delta_{g^0} P$ in (31).

Our model makes is plausible to interpret the measurements of $a_{f,g}$ as joint position-momentum measurements. This becomes most clear if we consider (strict) preparatory measurements. Within the present model the only possibility of strict preparatory measurements for arbitrary initial states $P(\varphi]$ [in (23)] seems to be given by means of the preparations

$$\begin{aligned} \Phi_1(X_1) &= (\sqrt{\pi} b)^{-1/2} \exp\left\{-\frac{1}{2b^2} X_1^2\right\} \\ \tilde{\Phi}_2(\Pi_2) &= \left(\frac{\sqrt{\pi}\hbar}{c}\right)^{-1/2} \exp\left\{-\frac{c^2}{2\hbar^2}\Pi_2^2\right\} \end{aligned} \qquad b \cdot c = \frac{\lambda\mu}{2}\hbar \tag{41}$$

This case is uniquely characterized by the minimal uncertainty product

$$(\Delta_f Q)^2 \cdot (\Delta_g P)^2 = \frac{\hbar^2}{4}$$

which corresponds to Gaussians $f_q(q') = |\varphi_{qp}(q')|^2$, $g_p = (p') = |\tilde{\varphi}_{qp}(p')|^2$. Arthurs and Kelly (1965) gave a treatment of just this special case by means of a Hamiltonian similar to (32) and with values $\lambda = \mu = \hbar = 1$, $b \cdot c = \frac{1}{2}$. Our model has been developed as a generalization of their example in order to trace back the origin of the UR to the dynamics of the measuring process.

It can be shown that in the general case our model fulfils condition (22) of (not strict) preparatory measurements. There exists a class $\mathcal{S}$ of initial states $P[\varphi]$ (obeying $\Delta_\varphi Q \gg \Delta_f Q$, $\Delta_\varphi P \gg \Delta_g P$ and thus $\Delta_\varphi Q \cdot \Delta_\varphi P \gg \hbar/2$) for which our model leads to final states W_{qp} characterized by (22). (The construction goes similar to that for single observables, see Section 3.1.)

The possibility of (strict) preparatory measurements allows us to interpret the unsharpnesses $\Delta_f Q$ and $\Delta_g P$ as measures of the *position and momentum indeterminateness* characteristic of the object after such measurements.

As we found for single real position measurements there are no strongly predictable joint measurements: P. Lahti (1981) has shown that, for arbitrary joint observables $a_{f,g}$ and arbitrary states W

$$p_W^{f,g}(\Delta) = \mathrm{tr}[W \cdot a_{f,g}(\Delta)] < 1 \qquad [\Delta \in \mathcal{B}(\mathbb{R}^2)]$$

a result resting on Theorem (6). But again it can be shown that observables $a_{f,g}$ admit weakly repeatable and thus weakly predictable measurements.

3.3. A New "Derivation" of the UR

Although the UR for f, g follows formally from the representation (40) of the density $\mathcal{F}(q, p)$ through some state operator it is instructive to prove it directly by means of (39). We put

$$\varepsilon \equiv (\Delta_f Q)^2, \qquad \eta \equiv (\Delta_g P)^2, \qquad \varepsilon_i \equiv (\Delta_{\Phi_i} Q_i)^2, \qquad \eta_i \equiv (\Delta_{\Phi_i} P_i)^2.$$

Now

$$\varepsilon \cdot \eta = \tfrac{1}{4}(\varepsilon_1 \cdot \eta_1 + \varepsilon_2 \cdot \eta_2) + \left(\frac{1}{\lambda^2 \mu^2} \varepsilon_1 \cdot \varepsilon_2 + \frac{\lambda^2 \mu^2}{16} \eta_1 \cdot \eta_2 \right)$$

The expressions

$$h_1 \equiv \tfrac{1}{4}(\varepsilon_1 \cdot \eta_1 + \varepsilon_2 \cdot \eta_2)$$

$$h_2 \equiv \frac{1}{\lambda^2 \mu^2} \varepsilon_1 \cdot \varepsilon_2 + \frac{\lambda^2 \mu^2}{16} \eta_1 \cdot \eta_2 = \frac{\hbar^2}{16} \left[\frac{16}{\hbar^2 \lambda^2 \mu^2} \varepsilon_1 \cdot \varepsilon_2 + \frac{\lambda^2 \mu^2}{\hbar^2} \eta_1 \cdot \eta_2 \right]$$

can be estimated by using the uncertainty relations for M_1, M_2, $\varepsilon_i \cdot \eta_i \geq \hbar^2/4$ $(i = 1, 2)$:

$$h_1 \geq \frac{\hbar^2}{8}, \qquad h_2 \geq \frac{\hbar^2}{16}\left(x + \frac{1}{x} \right) \geq \frac{\hbar^2}{8}, \qquad x \equiv \frac{16}{\hbar^2 \lambda^2 \mu^2} \varepsilon_1 \cdot \varepsilon_2 > 0$$

Thus the sum is

$$\varepsilon \cdot \eta = h_1 + h_2 \geq \frac{\hbar^2}{4}$$

The equality sign holds for $h_1 = h_2 = \hbar^2/8$, that is,

$$\varepsilon_i \cdot \eta_i = \frac{\hbar^2}{4}, \qquad x = 1, \qquad \text{i.e., } \varepsilon_1 = \frac{\lambda^2 \mu^2}{4} \eta_2$$

which is the case of the strict preparatory measurement (41). Each of the terms h_i would suffice to give an uncertainty relation

$$\varepsilon \cdot \eta \geq h_i \geq \frac{\hbar^2}{8}$$

the structure of h_i is such that the UR for each single one of the measuring systems M_i leads to

$$\varepsilon \cdot \eta \geq \frac{\hbar^2}{16}$$

It is this aspect of the UR which has been emphasized by Bohr: the impossibility of arbitrarily sharp simultaneous measurements results from the fact that the measuring device being a physical system is subject to the UR.

There is another important feature of the UR that can be studied within our model: as Heisenberg pointed out, the presence of a position-measuring device "disturbs" the sharpness of determination of momentum and vice versa. This becomes most clear from the second term h_2. The "undisturbed" product $(\varepsilon_1 \cdot \varepsilon_2)/(\lambda^2\mu^2)$ may be made arbitrarily small but then the product $\lambda^2\mu^2\eta_1 \cdot \eta_2/16$ of the "disturbing" contributions in (39) will increase to infinity. If one tries to diminish the ε_1, ε_2 the UR of the M_1, M_2 comes into play to raise the product $\eta_1 \cdot \eta_2$; similarly the interaction parameters λ, μ are placed in such a way that there is no possibility to escape the limit $h_2 \geq \hbar^2/8$.

3.4. Origin of the UR: Mutual Disturbance of the Measuring Devices

It has often been tried to explain the UR as a restriction of simultaneous measurability from an uncontrollable influence on the object system by the measuring act. That this view is somewhat misleading can be seen from our model: what is decisive is a mutual influence between the measuring instruments (cf. de Muynck et al., 1979). The commutation relation (3) gives rise to a "coupling" of the measuring particles which can be read off from (33c):

$$H_I^{(1,2)} = \frac{i}{\hbar}\frac{\lambda\mu}{2}[Q, P]\otimes P_1 \otimes P_2 \delta(t) = -\frac{\lambda\mu}{2} I \otimes P_1 \otimes P_2 \delta(t)$$

Obviously it is his term containing the product $\lambda \cdot \mu$ that is responsible for the disturbing unsharpnesses, so that one might try to introduce an artificial compensating interaction $-H_I^{(1,2)}$. It is clear, however, that there is no way to get rid of the UR.

Replacing (32) and (33c) by

$$H_I^{(n)} = \left\{\lambda Q \otimes P_1 \otimes I + \mu P \otimes I \otimes P_2 + \frac{i}{\hbar}\frac{\lambda\mu}{2} n[Q, P]\otimes P_1 \otimes P_2\right\}\delta(t)$$

$$U^{(n)} = \exp\left\{-\frac{i}{\hbar}\lambda Q \otimes P_1 \otimes I\right\} \cdot \exp\left\{-\frac{i}{\hbar}\mu P \otimes I \otimes P_2\right\}$$

$$\cdot \exp\left\{\frac{\lambda\mu}{2}(n+1)[Q, P]\otimes P_1 \otimes P_2\right\}$$

we get a measurement of an observable $a_{f^{(n)},g^{(n)}}$ with unsharpnesses

$$(\Delta_{f^{(n)}}Q)^2 = \frac{1}{\lambda^2}(\Delta_{\Phi_1}Q_1)^2 + (n-1)^2\frac{\mu^2}{4}(\Delta_{\Phi_2}P_2)^2$$

$$(\Delta_{g^{(n)}}P)^2 = \frac{1}{\mu^2}(\Delta_{\Phi_2}Q_2)^2 + (n+1)^2\frac{\lambda^2}{4}(\Delta_{\Phi_1}P_1)^2$$

for which the UR is still valid: all that one can do is to eliminate just one of the disturbing unsharpnesses $(n = \pm 1)$.

3.5. Order of Magnitude of Planck's Constant $\hbar$—Classical Limit

The concept of real measurements and its realization by our models enabled us to give a precise quantum mechanical reconstruction of the old semiclassical argumentations in favor of the individualistic interpretation of the UR. The interrelations between measurement unsharpnesses, mutual disturbance of measuring devices, and canonical commutation relations have become quite clear. Yet, up to now the order of magnitude of Planck's constant $\hbar$ was of no account; its precise value depends on the choice of units. As we shall see now the magnitude of $\hbar$ determines whether a joint measurement may be considered "classical" or not.

In classical physics a notion of "exactly defined" particle trajectories measurable with "arbitrary precision" is admissible which implies the following presuppositions:

(I) A particle possesses "exact" values of position and momentum.
(II) Measurements of position and momentum do not disturb each other.
(III) The "accuracies" of measurements may be increased arbitrarily.
(IV) The object system need not get influenced by the measurements.

We may readily give a quantitative formulation of (I)–(IV) in terms of our model if we identify the classical "inaccuracies" with the unsharpnesses of unsharp observables. As we have seen in equation (30) this identification is possible for a certain class of initial states φ obeying postulate (I) in the following form:

(I′) States φ entail position and momentum indeterminacies which are small compared to the measuring unsharpnesses:

$$(\mathrm{I}'1) \quad \Delta_\varphi Q \ll \Delta_f Q; \qquad (\mathrm{I}'2) \quad \Delta_\varphi P \ll \Delta_g P$$

Strictly speaking the functions $\sqrt{\lambda}\Phi_1(\lambda(q-q'))$ and $\sqrt{\mu}\Phi_2(\mu(p-p'))$ shall be slowly varying everywhere within ranges of q' and p' values for which $\varphi(q')$ and $\tilde{\varphi}(p')$ are appreciably different from zero. The above inequalities will be taken as symbolic expressions of these relationships.

Conditions (II)–(IV) are translated in a similar way.

(II′) The measuring unsharpnesses are practically identical with those of the single measurements:

$$\Delta_f Q \simeq \frac{1}{\lambda}\Delta_{\Phi_1} Q_1, \qquad \Delta_g P \simeq \frac{1}{\mu}\Delta_{\Phi_2} Q_2$$

that is,

$$\text{(II'1)}\quad \frac{1}{\lambda}\Delta_{\Phi_1}Q_1 \gg \frac{\mu}{2}\Delta_{\Phi_2}P_2; \qquad \text{(II'2)}\quad \frac{1}{\mu}\Delta_{\Phi_2}Q_2 \gg \frac{\lambda}{2}\Delta_{\Phi_1}P_1$$

As in (I′) these inequalities are assumed to hold in a strong sense such that the (translated) functions $\sqrt{\lambda}\,\Phi_1(\lambda q), \sqrt{\mu}\,\Phi_2(\mu p)$ are practically constant in intervals where $\tilde{\Phi}_2$ and $\tilde{\Phi}_1$ are not negligible.

(III′) The constant $\hbar^2/4$ is very small in comparison to the product of the unsharpnesses

$$(\Delta_f Q)^2 \cdot (\Delta_g P)^2 \gg \frac{\hbar^2}{4}$$

which means that it may be neglected as a lower limit.

(IV′) The state φ of the system is left practically unchanged during the measuring process.

Now it is easy to show that each of the conditions (I′) and (II′) implies (III′), whereas only the combination of both is sufficient for (IV′). (We do not give the proofs here.) Especially, from (I′) and (II′) we get the following. Let $q_0 = \langle Q\rangle_\varphi$, $p_0 = \langle P\rangle_\varphi$, then the change of state for initial states $P[\varphi]$ obeying (I′) and (II′) is

$$P[\varphi] \to \tilde{W}(E, F) \simeq P[\varphi]\left\{\int_E dq\,\lambda|\Phi_1(\lambda(q-q_0))|^2 \int_F dp\,\mu|\Phi_2(\mu(p-p_0))|\right\}^2$$

The normalized final state $W(E, F)$ is identical to the initial state $P[\varphi]$, that is, (IV′) holds.

Conditions (I′) and (II′) lead to measurements which possess all features typical of classical mechanical measuring situations, namely, (I) to (IV); since the values of position and momentum (q_0, p_0) are (almost) exactly defined (I) there exist weakly predictable results $E \times F$: the probability

$$\begin{aligned} p_\varphi^{f,g}(E\times F) &= \mathrm{tr}(\tilde{W}(E, F)] \\ &\simeq \int_E dq\,\lambda|\Phi_1(\lambda(q-q_0)|^2 \cdot \int_F dp\,\mu|\Phi_2(\mu(p-p_0))|^2 \end{aligned}$$

is near to unity if $E \supseteq E_0 = (q_0 - \delta q, q_0 + \delta q)$, $F \supseteq F_0 = (p_0 - \delta p, p_0 + \delta p)$ with sufficiently large δq, δp. Further, since the system does not get influenced (IV) results $E \supseteq E_0$, $F \supseteq F_0$ are weakly repeatable. One simply has to combine two position and momentum measuring instruments with a "macroscopic" product of the undisturbed unsharpnesses $\Delta_{f^0}Q$, $\Delta_{g^0}P$ (II′), then for initial states obeying (I′) we enter the classical situation ((III), (IV)).

Unsharpness distributions f, g may be interpreted as confidence functions in the classical case only. The probability for the registration of unsharp points E_q, F_p (with $\delta q \ll \Delta_f Q$, $\delta p \ll \Delta_g P$),

$$p_\varphi^{f,g}(E_q \times F_p) \simeq (\delta q\, \delta p)\lambda|\Phi_1(\lambda(q-q_0))|^2 \cdot \mu|\Phi_2(\mu(p-p_0))|^2$$

is nonnegligible only for values q, p close to q_0, p_0,

$$|q-q_0| \lesssim \Delta_f Q, \qquad |p-p_0| \lesssim \Delta_g P$$

Which result (q, p) *will* come about is *objectively undecided* as the unsharpnesses $\Delta_f Q \simeq (1/\lambda)\Delta_{\Phi_1} Q_1$, $\Delta_g P \simeq (1/\mu)\Delta_{\Phi_2} Q_2$ originate from some indeterminacy in the M_i states Φ_i. *If* a result (q^*, p^*) *has been established* the "true" values (q_0, p_0) are known only up to *subjective uncertainties* $\Delta_f Q$, $\Delta_g P$.

To sum up, the above considerations reveal the decisive role of Planck's constant. Only with respect to measuring instruments yielding macroscopic uncertainty products $\Delta_f Q \cdot \Delta_g P \gg \hbar$ is it possible to neglect certain restrictions incorporated in the quantum mechanical language and to make use of the language of classical physics.

It appears remarkable that the indeterminacy product $\Delta_\varphi Q \cdot \Delta_\varphi P$ need not be "macroscopic" for classical measurements. Thus one can measure classical trajectories for quantum mechanical, microscopic particles (S, φ) with $\Delta_\varphi Q \cdot \Delta_\varphi P \sim \hbar$. As a specific example we should mention the Wilson chamber (Heisenberg, 1930) where microscopic particles propagate along visible macroscopic paths. The collapse of wave packets caused by ionization of scattering molecules becomes negligible if the particle travels fast enough to avoid any appreciable spreading of the packets. Heisenberg gives a treatment of the UR for this example by means of Huygens' principle, and it becomes particularly clear that the notion of unsharp observables fits with his intuitive conception of "measuring inaccuracy".

3.6. Heisenberg's Slit Experiment

Although our model allows an utmost clear and mathematically simple analysis of what happens in joint measurements it must be admitted that we have been using quite an *unrealistic* Hamiltonian (32) for a model of *real* measurements. Therefore we shall conclude our work by discussing as an intuitive example Heisenberg's well known slit experiment (Figure 1). [Besides, an analysis of Wootters' and Zurek's (1979) treatment of Bohr's double slit experiment shows that it may be reformulated in terms of unsharp

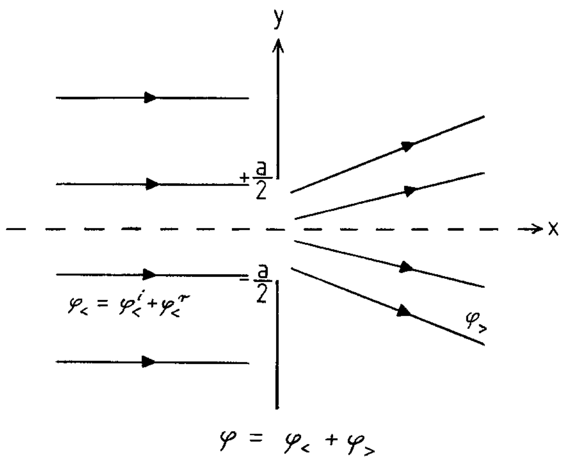

Fig. 1. Slit experiment.

observable, but we shall not enter this topic now.] The object system—for example, an electron—impinges on a wall at $x = 0$ which has a slit of width a (along the y axis). The electron is assumed to possess a rather well-defined momentum from a preparatory momentum measurement. Then we may represent the incident part $\varphi_<^i$ of the wave function $\varphi_<(x, y)$ as a plane wave with

$$\langle P_y\rangle_{\varphi_<^i} = 0, \qquad \Delta_{\varphi_<^i} P_x \simeq \Delta_{\varphi_<^i} P_y \ll \frac{2\pi\hbar}{a}, \qquad \Delta_{\varphi_>^i} Y \gg a$$

The wall will be described as an infinitely high potential well with negligible thickness. Then the stationary Schrödinger equation reduces to the classical optical wave equation where the slit can be treated as boundary conditions. This set-up may be interpreted as preparatory real position measurement. The Hilbert space of the (two-dimensional) electron is

$$\mathcal{H} = L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \equiv \mathcal{H}^{(x)} \otimes \mathcal{H}^{(y)}$$

thus "system $M \triangleq \mathcal{H}^{(x)}$" may be used for a position measurement on "system $S \triangleq \mathcal{H}^{(y)}$", the external potential $V(x, y)$ serving as "interaction H_I between S and M". By an ideal first-kind measurement of the observable

$$F(Q_M) = F(X) = \sum_{n=-\infty}^{\infty} x_n P^X(E_n)$$

$$E_n = (x_n - \delta x, x_n + \delta x], \qquad x_0 = 0, \qquad x_{n+1} = x_n + 2\delta x$$

it is possible to decide which of the orthogonal parts $\varphi_>$, $\varphi_<$ will be realized, that is, whether or not the particle has passed the slit. After registration of an E_n $(x_n > 0)$ the state of S & M is

$$[P^X(E_n)\otimes I]\varphi = [P^X(E_n)\otimes I]\varphi_>$$

whereas the state of S alone becomes

$$\Psi^{(n)}(y) \simeq \varphi_>(x_n, y)$$

(if we neglect the width $2\delta x$ of E_n).

Within the slit $(n=0, x_0=0)$ the function $\Psi^{(0)}(y)$ is approximately constant for wavelengths $\lambda \equiv (2\pi\hbar/\langle P_x\rangle_{\varphi_<^i}) \ll a$:

$$\Psi^{(0)}(y) \simeq a^{-1/2}\chi_{[-a/2,a/2]}(y) \qquad (\lambda \ll a) \tag{42}$$

[strictly speaking, $\Psi^{(0)}(y)$ tends to zero for $|y| \to a/2$; see Beck and Nussenzveig (1958)]. Thus the slit provides a preparation of an unsharp point

$$(y=0, f_0), \qquad f_0(y) \simeq a^{-1}\chi_{[-a/2,a/2]}(y) \tag{43a}$$

with position unsharpness

$$\Delta y = \Delta_{f_0} Y \simeq a \qquad \text{(up to a factor near to 1)} \tag{43b}$$

Momentum indeterminacy in $\Psi^{(0)}$ is represented by an infinite spread $\Delta_{\Psi^{(0)}} P_y = \infty$ which does not get finite even for an exact calculation of $\Psi^{(0)}$. Only a (nonsingular) potential well with finite thickness and smooth surface would lead to a finite $\Delta_{\Psi^{(0)}} P_y$ as pointed out by Beck and Nussenzveig. Following Heisenberg, within the present idealization (42) we take as a measure of momentum indeterminacy the width of the central peak in the interference pattern $|\Psi^{(n)}(y)|^2$ $(x_n \gg a \gg \lambda)$ which is essentially determined by the Fourier transform $\tilde{\Psi}^{(0)}(p_y)$:

$$\delta p_y \simeq \frac{2\pi\hbar}{a}, \qquad \langle P_y\rangle_{\psi^{(n)}} = 0 \tag{44}$$

The incident state was assumed to result from a preparatory momentum measurement, so it is characterized by an unsharp value $(p_y = 0, g_0)$, $\Delta_{g_0} P_y \simeq \Delta_{\varphi^i_<} P_y \ll 2\pi\hbar/a$. After the electron has passed the slit there remains a modified unsharp value $(p_y = 0, g_0^{(n)})$ with $g_0^{(n)}(p_y) \simeq |\tilde{\Psi}^{(0)}(p_y)|^2$ and $\Delta_{g_0^{(n)}} P_y \simeq \delta p_y \simeq 2\pi\hbar/a$.

From (43) and (44) we get an uncertainty relation

$$\Delta_{f_0} Y \cdot \Delta_{g_0^{(n)}} P_y \simeq \Delta y \cdot \delta p_y \simeq 2\pi\hbar$$

which says—in full agreement with Heisenberg (1930)—that the presence of a position-measuring apparatus "disturbs" the "accuracy" of determination of momentum. This model, however, does not provide an operational interpretation of momentum unsharpness δp_y but only of position uncertainty $\Delta y \simeq a$.

4. CONCLUSION

In the present paper we provide an operational interpretation of the uncertainty relation for position and momentum referring to single physical systems. The strong preparatory property of first-kind measurements necessitates the introduction of unsharp observables in order to get a concept of joint measurements of incompatible observables (Section 2). The concept of unsharp observables is interpreted within a quantum mechanical theory of measurement proving it reasonable to speak of joint position and momentum measurements (Section 3).

Results of joint position-momentum measurements on an individual system may be unsharp points $((q, f_q), (p, g_p))$ with uncertainty distributions f_q, g_p centered around q, p and obeying the UR $\Delta_f Q \cdot \Delta_g P \geq \hbar/2$. This relation determines the lower limit of (individual) unsharpnesses of measuring results which are necessarily connected with joint measurements.

By means of several quantum mechanical models of unsharp position and of position-momentum measurements we give operational expressions of the unsharpnesses $\Delta_f Q$, $\Delta_g P$ in terms of quantities characterizing measuring interactions and preparations of measuring devices. From their origin the unsharpnesses may be interpreted as *measures of indeterminacy.* On the other hand, the significance of the $\Delta_f Q$, $\Delta_g P$ for the measured object depends on the circumstances. First, a certain class of initial states φ of the object admits (nonobjectifying) preparatory measurements; then the unsharpnesses $\Delta_f Q, \Delta_g P$ stand for *position and momentum indeterminatenesses of the object* after measurement. Second, in some "*classical*" *limit* with states φ representing well defined position q_0 and momentum p_0 $(\hbar/2 \ll \Delta_\varphi Q \cdot \Delta_\varphi P \ll \Delta_f Q \cdot \Delta_g P)$, $\Delta_f Q$ and $\Delta_g P$ may be regarded as measures of *subjective uncertainty* (ignorance) with respect to the "true" values q_0, p_0.

The model enables us to give a strictly quantum mechanical operational derivation of the uncertainty relation which serves as a reconstruction of the well-known semiclassical derivations.

Finally, as an intuitive example, we give a quantum measurement theoretical treatment of Heisenberg's slit experiment which shows—similarly to the former model—that it is the mutual disturbance of the measuring devices which is responsible for the occurrence of the UR for incompatible observables.

To sum up, what has been shown is the following: there are in quantum mechanics measurement situations to which the individualistic interpretation of the UR is applicable. In case of "real" joint measurements the measuring uncertainties are subject to an UR. The statistical interpretation remains valid with respect to sharp (first-kind) measurements.

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